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# BRANCHING PROCESSES IN RANDOM TREES

by

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# ABSTRACT


YANJMAA JUTMAAN. Branching processes in random trees.
(Under the direction of DR. STANISLAV MOLCHANOV)

We study the behavior of branching process in a random environment on trees in the critical, subcritical and supercritical case. We are interested in the case when both the branching and the step transition parameters are random quantities. We present quenched and annealed classifications for such processes and determine some limit theorems in the supercritical quenched case. Corollaries cover the percolation problem on random trees. The main tool is the compositions of random generating functions.

## ACKNOWLEDGMENTS

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## 1.1 Introduction

Branching Stochastic processes [23] are very interesting for mathematicians and physicists. They describe a multitude of phenomena from chain reactions to population dynamics. [20]. Random walks in random environment on $Z^d$ constitute one of the basic models of random motions in a random medium; see Bolthausen [2], Hughes [15], Molchanov [21] and Zeitouni [27], [28]. Several authors have considered branching random walk in a random environment [18], [1], [3]- [6], [8], [15], [10] and [14].

There is significant literature on the random walk and the branching (reaction diffusion) process in the random environment on the lattice $Z^d$ or the Euclidean space $R^d$, $d \geq 1$.

In most of these papers the classical Fourier analysis and the spectral theory of the Schröndinger type operators play the central role (explicitly and implicitly) corresponding to models and construction on the different graphs, manifolds and groups. To the best of our knowledge, branching random walks in a random environment on trees have not yet been considered in the literature [18], [17]. The goal of this paper is the analysis of random processes in the random environments on the simplest graphs: the trees. We study branching processes and random walks in the situation when transition probabilities are random and homogeneous. The paper is organized as follows: in this chapter we begin by introducing some notations and some preliminary results as well as introducing basic examples and models. In Chapter 2 we will define the Galton-Watson process in homogeneous medium, and its moment generating functions and moments. We discuss some limit theorems and show some proof.

In Chapter 3 We will introduce construction of random environment on the tree we prove several results on the annealed and quenched behavior of the Galton-Watson process $n(t, \omega, \omega_m)$. We also discuss some limit theorems. In Chapter 4 we study Percolation problem on the random tree, critical probability of percolation and some properties. Ordinary [9] and multiscale [19] percolation play a critical role in many applications [20].

## 1.2 Simple Galton-Watson process

Galton-Watson processes were introduced by Francis Galton in 1889 as a simple mathematical model for the propagation of family names. Galton-Watson processes continue to play a fundamental role in both the theory and applications of stochastic processes.

First, a description: A population of individuals (which may represent people, organisms, free neutrons, etc., depending on the context) evolves in discrete time $t = 0, 1, 2, \ldots$ according to the following rules. Each $n$th generation individual produces a random number (possibly 0) of individuals, called *offspring*, in the $(t+1)^{th}$ generation. The offspring counts $n(t, \omega)$ for distinct individual are mutually independent, and also independent of the offspring $X_{t,i}$ counts of individuals $t, i$ from earlier generations. Furthermore they are identically distributed. The state $n(t, \omega)$ of the Galton-Watson process at time $t$ is the number of individuals in the $t^{th}$ generation.

$$n(t+1, \omega) = \sum_{i=1}^{n(t,\omega)} X_{t,i}(\omega), \tag{1.1}$$

where $X_{t,i}$ are independent identically distributed (i.i.d) random variables on some probability space $(\Omega, \Gamma, P)$.

### 1.2.1 Generating function of $n(t, \omega)$

$$\varphi(z) = Ez^{X_{n,i}} \tag{1.2}$$

and

$$\varphi'(z)|_{z=1} = a = EX. \; EX(X-1) = a_{[2]}$$

The iterated of the generating function $\varphi_s$ will be defined by

$\varphi_0 = z$, $\varphi_{[1]} = \varphi(z)$,

$$\varphi_{[s+1]}(z) = \varphi(\varphi_{[s]}(z)) = \varphi_{[s]}(\varphi(z)), n = 1, 2, \dots \tag{1.3}$$

The following basic result was discovered by Watson (1874) and has been rediscovered a number of times since. See for instance Theorem 4.1 in [13],

Theorem 1.1. *The generating function of $n(t,\omega)$ is the $t$-th iterate $\varphi_{[t]}(z)$.*

### 1.2.2 Moments of the $n(t,\omega)$

We can obtain the moments of $n(t,\omega)$ by differentiating (1.3) at $z = 1$. Thus differentiating (1.3) yields

$$\varphi'_{[t+1]}(1) = \varphi'_{[t]}\left(\varphi(1)\right)\varphi'(1) \tag{1.4}$$

whence by induction $\varphi'_{[t]}(1) = a^t$ $, n = 0, 1, \dots$. If $\varphi''(1) < \infty$, we can differentiate (1.3) again, obtaining,

$$\varphi''_{[t+1]}(1) = \varphi'_{[t]}(1)\varphi''(1) + \varphi''_{[t]}(1)(\varphi')^2 \tag{1.5}$$

we obtain $\varphi''_{[t]}(1)$ by repeating application of (1.5) with $n = 0, 1, \dots$ ; thus

$$E(n_t(n_t - 1)) = \begin{cases} \frac{a_{[2]}a^t(a^t-1)}{a(a-1)} & , a \neq 1 \\ t \cdot a_{[2]} & , a = 1 \end{cases} \tag{1.6}$$

Theorem 1.2. *Expected value of* $E\{n(t,\omega)\}$ *is* $a^t$, $t = 0, 1, \ldots$. *If second factorial moment of* $X$ *is bounded, the the second factorial moment of* $n(t,\omega)$ *is given by* (1.6).

If higher moments of $X$ exist, then higher moments of $n(t,\omega)$ can be found in a similar fashion.

We now consider the problem originally posed by Galton: find the probability of extinction of a family.

Definition 1.1. [13] By extinction we mean the event that the random sequence $n(t,\omega)$ consists of zeros for all but a finite number of values of $t$.

Since $n(t,\omega)$ is integer-valued, extinction is also the event that $n(t,\omega) \to 0$. Moreover, since $P(n(t+1,\omega) = 0 | n(t,\omega) = 0) = 1$, we have the equalities

$$\begin{aligned} P(n(t,\omega) \to 0) &= P(n(t,\omega) = 0 \text{ for some } t) = P\left((n(1,\omega)=0) \cup (n(2,\omega)=0) \cup \ldots\right) \\ &= \lim_{t\to\infty} P\left((n(1,\omega)=0) \cup (n(2,\omega)=0) \cup \ldots (n(t,\omega)=0)\right) \\ &= \lim_{t\to\infty} P(n(t,\omega)=0) = \lim \varphi_{[t]}(0) \qquad (1.7) \end{aligned}$$

For simplicity of notation $n(t,\omega) = n(t)$.

Definition 1.2. Let $z*$ be the probability of extinction, i.e.,

$$z* = P(n(t) \to 0) = \lim \varphi_{[t]}(0) \qquad (1.8)$$

Theorem 1.3. *If* $a = EX \leq 1$, *the extinction probability* $z*$ *is 1. If* $a > 1$, *the extinction probability is the unique nonnegative solution less than 1 of the equation*

$$z = \varphi(z) \qquad (1.9)$$

This theorem was firs proved in complete generality by Steffensen (1930,1932)

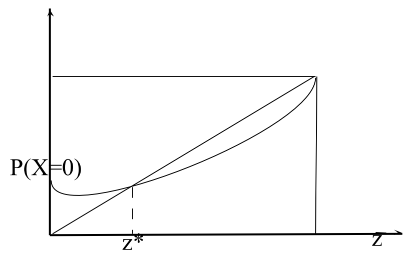


Figure 1.1: Moment generating function and extinction probability
Moment generating function $\varphi(z) = Ez^x$
where $z*$ is the solution of $z = \varphi(z)$ equation.

Solution and moment generating function are illustrated in Figure 1.1.

# CHAPTER 2: CLASSICAL GALTON-WATSON BRANCHING PROCESSES. DIFFERENT APPROACHES TO THE LIMIT THEOREM IN THE SUPERCRITICAL CASE

## 2.1 Limit Theorems

In next chapter we will prove several limit theorems for Branching process in random environment. To understand the idea of the method let us recall the classical result by N. A. Dmitriev and A.N. Kolmogorov [7]. Assume that $n(t, \omega)$, $t = 0, 1, \ldots,$ $n(1) = 1$ is the classical Galton-Watson process in the fixed environment.
If $\varphi(z) = Ez^X$ where $X$ is the number of the offspring of single particle; then

$$\varphi_{[n]}(z) = \varphi(\varphi(\ldots(\varphi(z))\ldots)) \tag{2.1}$$

$$E\{n(t, \omega)\} = a^t, \ a = a_1 = \varphi(1) = EX \tag{2.2}$$

Then, for $a > 1$, the supercritical case we have

Theorem 2.1. *( N.Dmitriev, A.Kolmogorov, [7])*

$$P\left(\frac{n(t, \omega)}{a^t} < x\right) \to G(x) \tag{2.3}$$

*where $G$ is a limit distribution.*

*Remark.* For continuous time branching Galton-Watson processes when the intensity of the duplication of a particle is $\lambda$, mortality rate is $\mu$, $\lambda > \mu$, then the limit law for $\frac{n(t,\omega)}{a^t}$ also can be found explicitly. This limit law is the exponential distribution with additional atom at $x = 0$.

$G$ is only known in the case of geometric distribution of $X$..

Consider the mass function of the geometric distribution

$$P\{X=0\}=\alpha, P\{X=1)=\{1-\alpha\}q\ ,\ \ldots\ P\{X_t=k\}=(1-\alpha)qp^{k-1},\ldots \qquad (2.4)$$

Then

$$\varphi(z)=Ez^{X_\tau}=\alpha+\frac{(1-\alpha)qz}{1-pz}=\frac{\alpha+z(q-\alpha)}{1-pz}$$
$$\varphi'(z)|_{z=1}=\frac{1-\alpha}{q}$$

The expansion of this generating function into Tailor series gives the geometric law for $k\geq 1$ and different probability for $\{X=0\}$, $P\{X=0\}=\alpha$. The composition $\varphi_{[n]}(z)=\varphi(\varphi(\ldots(\varphi(z))\ldots))$ has the same nature.

Note that

$$a=EX=\frac{1-\alpha}{q}$$

i.e process is supercritical for

$$\frac{1-\alpha}{q}>1$$
$$\Rightarrow 1>\alpha+q$$
$$\Rightarrow p>\alpha$$

The limiting distribution for $\frac{n(t,\omega)}{a^t}$, $t\to\infty$ is the combination of the exponential density and the atom at $x=0$, i.e it has density

$$\frac{n(t,\omega)}{a^t}\to g_\infty(x)=\beta\delta_0(s)+(1-\beta)\,\lambda e^{-\lambda x} \qquad (2.5)$$

Let's find the $\beta,\ \lambda$. First

$$1 = E\frac{n(t,\omega)}{a^t} \longrightarrow \int_0^\infty xg_\infty(x)dx = \frac{1-\beta}{\lambda}$$

i.e $\lambda = 1-\beta$. Secondly

$$P\{n(t,\omega)=0\} \xrightarrow[t\to\infty]{} s^* = \beta,$$

where $s^* = \varphi(s^*)$, $s^* < 1$. i.e

$$\begin{aligned}
s^* &= \frac{\alpha + s^*(q-\alpha)}{1-ps^*} \\
s^* - p(s^*)^2 &= \alpha + s^*(q-\alpha) \\
p(s^*)^2 - s^*(1-q+\alpha) + \alpha &= 0 \\
p(s^*)^2 - s^*(p+\alpha) + \alpha &= 0
\end{aligned}$$

One of the roots of this quadratic equation is $s^* = 1$, the second root which we need is

$$s^* = \frac{\alpha}{p} \tag{2.6}$$

Finally for $\beta, \lambda$ we have the linear system

$$\begin{cases} \lambda &= 1-\beta \\ \alpha &= p\beta \end{cases} \Rightarrow \begin{cases} \lambda &= 1-\beta = \frac{p-\alpha}{p} \\ \beta &= \frac{\alpha}{p} < 1 \end{cases}$$

Let us return to our main direction and prove (2.3). Our proof will be different from the original work of Dimitriev and Kolmogorov.

*Proof.* 1 (Martingale approach).

Consider $\xi(t) = \frac{n(t,\omega)}{a^t}$ and $F_{\leq t} = \sigma(n(1), n(2), \ldots, n(t))$

Let

$$n(t+1) = \sum_{i=1}^{n(t,\omega)} X_{t,i},$$

Where $X_{t,i}$ are independent identically distributed random variable with generating function $\varphi(z)$

It is easy to see that $(n(t,\omega), F_{\leq t})$ is a martingale. In fact

$$E\{n(t+1)|F_{\leq t}\} = n(t,\omega) \cdot a$$

So

$$E\{\xi(t+1)|F_{\leq t}\} = \frac{n(t,\omega)}{a^t} = \xi(t)$$

Due to Doob martingale convergence theorem $P$ almost surely there exist

$$\lim_{t\to\infty} \xi(t) = \xi(\infty)$$

and the law of the $\xi(\infty)$is $G(x)$ *Q.E.D.*

*Remark.* The same proof works, of course, in the case $a \leq 1$, but here the limit is equal to 0.

This method can not give any information about $G(\cdot)$. Let us try to give some more information about $G(x)$ distribution about moments or tail behavior. In order to tell that we need to define additional theorems and lemmas. One approach is a direct method using generating function and moment analysis other one is functional equational approach.

## 2.2 Moment of analysis approach to the limit theorem

The second method is based on the moments analysis.

Lemma 2.2. *Assume that*

$$P\{X = k\} \leq cq^k,\ k \geq 0;\ 0 < q < 1 \tag{2.7}$$

*then for any* $1 > q_1 > q$, *appropriate constant* $M_1(q_1)$ *and* $n \geq 1$,

$$a_{[n]} = EX(X-1)\cdot \ldots \cdot (X-n+1) \leq M_1 \cdot n! \cdot q_1^n, \quad n \geq 1 \tag{2.8}$$

*Proof.* Lemma (2.2) Let's take arbitrary $\Lambda < \frac{1}{q}$, i.e $\Lambda q < 1$ and put $M = \max\limits_{|z|=\Lambda} \varphi(z)$ Due to Cauchy theorem for $|z| = \Lambda$

$$\varphi(z) = \frac{1}{2\pi i} \oint_{|z|=\Lambda} \frac{\varphi(z) d\lambda}{z - \lambda} \tag{2.9}$$

Then

$$\varphi^{(n)}(z) = \frac{1}{2\pi i} \oint_{|z|=\Lambda} \frac{\varphi(z) n!}{(z-\lambda)^{n+1}} d\lambda \tag{2.10}$$

i.e for $z = 0$

$$\varphi^{(n)}(0) = EX(X-1)\cdot \ldots \cdot (X-n+1) = a_{[n]} \leq \frac{M \cdot n!}{\Lambda^n}, \quad n \geq 1 \tag{2.11}$$

In other term for arbitrary $q_1 > q$ and appropriate $M_1 = M(q_1) < \infty$

$$a_{[n]} = EX(X-1)\cdot \ldots \cdot (X-n+1) \leq M_1 \cdot n! \cdot q_1^n, \quad n \geq 1 \tag{2.12}$$

*Q.E.D.*

Let us define new theorem.

Theorem 2.3. *Under the assumptions of Lemma* (2.2) *for any* $k \geq 1$ *and* $t \to \infty$

$$\frac{En(t,\omega)(n(t,\omega)-1)\cdot \ldots \cdot (n(t,\omega)-k+1)}{a^{kt}} \to \nu_k \tag{2.13}$$

*and* $\nu_k \leq C_1 k! C^k$ *for appropriate* $C_1, C > 0$. *Calculation of the limiting moments can be done recursively.*

Corollary 2.4. *Due to well-known Carleman result, the moment problem ( reconstruct the distribution* $G(\cdot)$ *given moments* $\nu_k, k \geq 1$*) under the condition* $\nu_k \leq C_1 k! C^k$ *has unique solution and the Laplace transform*

$$\int_0^\infty e^{-\lambda x} dG(x) = \widehat{G}(\lambda)$$

*has analytic continuation at domain* $Re\lambda \geq -\delta$, $\delta > 0$ *( for appropriate* $\delta > 0$*), i.e distribution function* $G(x)$ *has an exponential tail.*

*Proof.* (of Theorem 2.3)

Let us use equation 1.2

$$\varphi_{[t]}(z) = \underbrace{\varphi(\varphi(\ldots\varphi}_{t}(z))\ldots) = \varphi_{[t-1]}(\varphi(z))$$

First,

$$\varphi_t'(1) = a_{[1]}(t) = a^t,\ a > 1, \text{where } \varphi'(1) = a \tag{2.14}$$

Then

$$\frac{E\{n(t,\omega)\}}{a^t} = 1 \xrightarrow[t\to\infty]{} \nu_1 = 1$$

Second,

$$\varphi_t''(1) = a_{[2]}(t) = \varphi_{[t-1]}''(\varphi')^2 + \varphi_{[t-1]}'\varphi''$$
$$= a_{[2]}(t-1)a^2 + a^{t-1}a_{[2]}(1) \tag{2.15}$$

Then

$$\begin{aligned} \frac{a_{[2]}(t)}{a^{2t}} &= \frac{a_{[2]}(t-1)}{a^{2(t-1)}} + \frac{a_{[2]}(1)}{a^{t+1}} \\ &= \frac{a_{[2]}(1)}{a^{t+1}} + \frac{a_{[2]}(1)}{a^{t}} + \frac{a_{[2]}(1)}{a^{t-1}} + \ldots + \frac{a_{[2]}(1)}{a^{2}} \\ &= a_{[2]}(1)\left(\frac{1}{a^2} + \frac{1}{a^3} + \ldots + \frac{1}{a^{t+1}}\right) \end{aligned} \tag{2.16}$$

It gives

$$\lim_{t\to\infty} \frac{E\{n(t,\omega)(n(t)-1)\}}{a^{2t}} = \lim_{t\to\infty} \frac{E\{n(t)^2\}}{a^{2t}} = \frac{a_{[2]}(1)}{a^2(1-\frac{1}{a})} = \frac{a_{[2]}(1)}{a(a-1)} \tag{2.17}$$

Assume that we already proved that for $l \leq k-1$

$$\frac{a_{[l]}(t)}{a^{lt}} = \frac{E\{n(t)(n(t)-1)\ldots(n(t)-l+1)\}}{a^{lt}} \leq C_0 l! C_1^l$$

and select such $C_0, C_1$ that recursively we can reconstruct the same estimate for $l = k$

Meanwhile, a higher order version of the chain rule (see e.g. [11] p. 33) states that

$$f^{(s)}(g) = \sum_{m_1,m_2,\ldots,m_s} \frac{s!}{m_1!m_2!\cdots m_s!}[f^{(m_1+m_2+\cdots+m_s)} \prod_{j=1}^{s} \left(\frac{g^{(j)}}{j!}\right)^{m_j} \tag{2.18}$$

where $1 \cdot m_1 + \ldots + n \cdot m_n = n$

$$
\begin{aligned}
&\varphi_{[t]}^{(k)}(z)|_{z=1} = a_{[k]}(t) \\
&= \sum_{m_1,m_2,\ldots,m_k} \frac{k!}{m_1!m_2!\cdots m_k!}\left[\varphi_{t-1}^{(m_1+m_2+\cdots+m_k)} \prod_{j=1}^{k}\left(\frac{\varphi^{(j)}(1)}{j!}\right)^{m_j}\right] \\
&= \varphi_{t-1}^{(k)}(1) a_{[1]}^{k} + \\
&+ \sum_{\substack{m_1,\, m_2,\, \ldots,\, m_k \\ m_1 \neq k}} \frac{n!}{m_1!m_2!\cdots m_k!}\left[\varphi_{t-1}^{(m_1+m_2+\cdots+m_k)} \varphi_{t-1}' \prod_{j=1}^{k}\left(\frac{\varphi_t^{(j)}}{j!}\right)^{m_j}\right] \\
&+ \sum_{1\cdot m_1+\ldots+k\cdot m_{k-1}=k} \frac{k!}{1!^{m_1} m_1! 2!^{m_1} m_2! \cdots (k)!^{m_1} m_{k-1}!} \times \\
&\times \left[\varphi_{t-1}^{(m_1+m_2+\cdots+m_{k-1})}\left(\sum_{j=1}^{n} m_j \frac{\varphi_t^{(j+1)}}{\varphi_t^{(j)}} \prod_{s=1}^{k-1}\left(\varphi_t^{(s)}\right)^{m_j}\right)\right]
\end{aligned}
$$

where $1 \cdot m_1 + \ldots + k \cdot m_k = k$ and highest order derivative in the last two sums is $k-1$ and $\varphi^{(j)}(1) \leq M_1 q_1^j j!$ and by the inductive assumption

$$\varphi_{[t-1]}^{(l)}(1) \leq C_0 \cdot l! \cdot C_1^l \cdot a^{l(t-1)}$$

Substitution of these inequalities in the long formula above gives, after some calculation $\varphi_{[t-1]}^{(l)}(1) \leq C_0 \cdot l! \cdot C_1^l \cdot a^{l(t-1)}$.

*Q.E.D.*

Let me now give more calculation of higher order differential equation (2.18).

Let us use equation 1.2 again

$$\varphi_{[t]}(z) = \underbrace{\varphi(\varphi(\ldots\varphi}_{t}(z))\ldots) = \varphi_{[t-1]}(\varphi(z))$$

Let us introduce new notation

$$A^{(n)}(t) = \varphi_t^{(n)}(z)|_{z=1} = En(t)(n(t)-1)(n(t)-n+1) \tag{2.19}$$

$$A^{(n)}(1) = a^{[n]} = \varphi^{(n)}(1) = EX(X-1)(X-2)\dots(X-n+1) \tag{2.20}$$

$a = a^{[1]} = EX > 1$, $X$ is the number of the offsprings of a single particle. The central recursive relation

$$\begin{aligned} A^{(n)}(t) &= \frac{\partial}{\partial z^n}\varphi_{[t-1]}(\varphi(z))|_{z=1} \\ &= \sum \underbrace{\frac{n!}{i!j!h!\cdots k!}}_{lfactor} \underbrace{\varphi_{[t-1]}^{(m)}}_{A^{(m)}(t-1)} \left(a^{[1]}\right)^i \left(\frac{a^{[2]}}{2!}\right)^j \left(\frac{a^{[3]}}{3!}\right)^h \cdots \left(\frac{a^{[l]}}{l!}\right)^k \end{aligned} \tag{2.21}$$

Summation over $i, j, \dots, k$ such that

$$\begin{cases} i + 2j + 3h + \dots + lk = n \\ i + j + h + \dots + k = m \end{cases} \Rightarrow \begin{cases} j + 2h + \dots + (l-1)k = n - m \\ i + j + h + \dots + k = m \end{cases}$$

Several first terms

Case I. $n = 1$, $m = 1$, $i = 1$

$$A^{[1]}(t) = \frac{1!}{1!}A^{[1]}(t-1)\cdot a^{[1]} \Rightarrow A^{[1]}(t) = En(t) = a^t$$

Case II. $n = 2$

$$\begin{cases} i + 2j = 2 \\ i + j = m \end{cases} \Rightarrow \begin{cases} m = 2, i = 2, j = 0 \\ m = 1, i = 1, j = 1 \end{cases}$$

$$A^{[2]}(t) = \frac{2!}{2!0!}A^{[2]}(t-1)\cdot a^2 + \frac{2!}{1!0!}A^{[1]}(t-1)\cdot \frac{a^{[2]}}{2!}$$

$$A^{[2]}(t) = A^{[2]}(t-1) \cdot a^2 + a^{t-1} \cdot a^{[2]}$$

$$\begin{aligned}
\tfrac{A^{[1]}(t)}{a^{2t}} &= \tfrac{A^{[2]}(t-1)}{a^{2(t-1)}} + \tfrac{a^{[2]}}{a^{t+1}} \\
&= \tfrac{A^{[2]}(t-2)}{a^{2(t-2)}} + \tfrac{a^{[2]}}{a^{t+1}} + \tfrac{a^{[2]}}{a^{t}} \\
&= a^{[2]}\left(\tfrac{1}{a^2} + \tfrac{1}{a^3} + \cdots + \tfrac{1}{a^{t+1}}\right)
\end{aligned}$$

Case III. $n = 3$

$$\begin{cases} i + 2j + 3h = 3 \\ i + j + h = m \end{cases} \Rightarrow \begin{cases} m = 1, i = 0, j = 0, h = 1 \\ m = 2, i = 1, j = 1, h = 0 \\ m = 3, i = 3, j = 0, h = 0 \end{cases}$$

$$A^{[3]}(t) = \frac{3!}{3!0!0!} A^{[3]}(t-1) \cdot a^3 + \frac{3!}{1!1!0!} A^{[2]}(t-1) \cdot a \cdot a^{[2]} + \frac{3!}{1!0!0!} A^{[1]}(t-1) \cdot \frac{a^{[3]}}{3!}$$

$$\frac{A^{[3]}(t)}{a^{3t}} = \frac{A^{[3]}(t-1)}{a^{3(t-1)}} + 3\frac{A^{[2]}(t-1)}{a^{2(t-1)}} \cdot \frac{a^{[2]}}{a^{t+1}} + \frac{a^{[3]}}{a^{2t+1}}$$

We now give another approach to calculating moments. This approach works under the assumption that we have already proved Theorem 2.1.

Recall the number $n(1)$ is the number of particles at time $t = 1$. Basically we will have $n(1)$ trees. This is illustrated in Figure 2.1

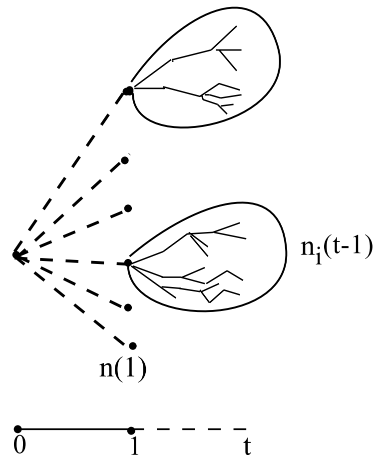


Figure 2.1: Forest and $n(1)$ trees
Where $n_i(t-1)$ is $i^{\text{th}}$ tree

Note that

$$\begin{aligned}\widetilde{\varphi}_t(\lambda) &:= Ee^{-\frac{\lambda n(t)}{a^t}} = Ee^{-\frac{\lambda}{a}\sum_{i=0}^{n(1)}\frac{n_i(t-1)}{a^{t-1}}} \\ &= E\left[E\left(e^{-\frac{\lambda}{a}\sum_{i=0}^{n(1)}\frac{n_i(t-1)}{a^{t-1}}}|n(1)\right)\right] = \widetilde{\varphi}\left(\widetilde{\varphi}_{t-1}\left(\tfrac{\lambda}{a}\right)\right)\end{aligned} \tag{2.22}$$

and by Theorem 2.1

$$\widetilde{\varphi}_t(\lambda) = Ee^{\lambda\frac{n(t)}{a^t}} \xrightarrow[t\to\infty]{} \widetilde{\psi}(\lambda) \tag{2.23}$$

Limiting Laplace transform satisfies the equation since we already proved the convergence using Martingale method.

$$\widetilde{\psi}(\lambda) = \widetilde{\varphi}_X\left(\widetilde{\psi}\left(\frac{\lambda}{a}\right)\right) \tag{2.24}$$

This is the functional equation for

$$\widetilde{\psi}(\lambda) = \int_0^\infty e^{-\lambda x} dG(x)$$

where $G(x) = \lim_{t \to \infty} P\left(\frac{n(t)}{a^t} < x\right)$

We will now use this this to calculate the first few moments of the limiting distribution and show that they are the same as those given by previous method.

$$\widetilde{\psi}'(\lambda)|_{\lambda=0} = En^* = 1 \text{ where } n^* = \frac{n(t)}{a^t} \tag{2.25}$$

$$\widetilde{\psi}'(\lambda)|_{\lambda=0} = \widetilde{\varphi}'\left(\widetilde{\psi}\left(\frac{\lambda}{a}\right)\right)\frac{1}{a}\widetilde{\psi}'\left(\frac{\lambda}{a}\right)|_{\lambda=0} = \widetilde{\varphi}'(1)\frac{1}{a} = 1$$

$$\begin{aligned}
\widetilde{\psi}''(\lambda)|_{\lambda=0} &= \widetilde{\varphi}''\left(\widetilde{\psi}\left(\tfrac{\lambda}{a}\right)\right)\tfrac{1}{a^2}\left(\widetilde{\psi}'\left(\tfrac{\lambda}{a}\right)|_{\lambda=0}\right)^2 + \widetilde{\varphi}'\left(\widetilde{\psi}\left(\tfrac{\lambda}{a}\right)\right)\tfrac{1}{a^2}\widetilde{\psi}''\left(\tfrac{\lambda}{a}\right)|_{\lambda=0} \\
&= \widetilde{\varphi}''(1)\tfrac{1}{a^2}\left(\widetilde{\psi}'(0)\right)^2 + \widetilde{\varphi}'(1)\tfrac{1}{a^2}\widetilde{\psi}''(0) \\
&= \tfrac{a_{[2]}}{a^2} + \tfrac{1}{a}\widetilde{\psi}''(0)
\end{aligned}$$

$$\widetilde{\psi}''(0)\left(1 - \frac{1}{a}\right) = \frac{a_{[2]}}{a^2}$$

$$\widetilde{\psi}''(0) = E(n^*)^2 = \frac{a_{[2]}}{a(a-1)} \tag{2.26}$$

The functional equational approach will give easier calculation for moments than direct moment calculation approach.

# CHAPTER 3: GALTON-WATSON IN RANDOM ENVIRONMENT

## 3.1 Construction of Random Environment

Let us now introduce the Galton-Watson process in the random environment. For classical Galton-Watson branching processes it is assumed that individuals reproduce independently of each other according to some given offspring distribution. In the setting of this paper the offspring distribution varies in a random fashion, independently from one generation to the other. A mathematical formulation of the model is as follows. We replace the term $n(t,\omega)$ in the equation (1.1) by $n(t,\omega,\omega_m)$. Here the random variable $n(t,\omega,\omega_m)$ belongs to a new probability space ($\Omega_m \times \Omega$, $\Gamma_m \times \Gamma, P \times P_m$). More precisely we have a new model defined as:

Definition 3.1. Let $n(t,\omega,\omega_m)$ be the number of the particles at the moment $t = 0, 1, \ldots$. Then

$$n(t+1,\omega,\omega_m) = \sum_{i=1}^{n(t,\omega,\omega_m)} X_{t,i}(\omega,\omega_m), \tag{3.1}$$

where $X_{t,i}(\omega,\omega_m)$ are i.i.d random variables whose distribution depends on $t$ and selected independently for different generations.

For simplicity let us write $X_{\cdot}(\omega,\omega_m) = X_{\cdot}(\omega_m)$

Let me give you several examples related to (3.1).

Example 3.1. Let

$$X_{t,\cdot} = \begin{cases} 2 & \text{with probability } \; p_t(\omega_m) \\ 0 & \text{with probability } \; 1-p_t(\omega_m) \end{cases}$$

and $p_t$ are i.i.d. random variables with values on $(0,1)$ and same density $f(p)$, $p \in (0,1)$

Moment generating function becomes

$$\varphi_t(z,\omega_m) = p_t(\omega_m) + (1 - p_t(\omega_m))z^2$$

Example 3.2. Let

$$p_0(\omega_m) + p_1(\omega_m) + \ldots + p_l(\omega_m) = 1$$

be a symplex in $R^{l+1}$ and $f(p)dp$ is density on this sympex. Let $\overrightarrow{P_t}$ be i.i.d vectors with the density f(p) and

$$X_t = \begin{cases} 0 & \text{with probability } p_{t,0}(\omega_m) \\ 1 & \text{with probability } p_{t,1}(\omega_m) \\ \vdots & \vdots \\ l & \text{with probability } p_{t,l}(\omega_m) \end{cases}$$

then moment generating function become

$$\varphi_t(z,\omega_m) = p_{t,0}(\omega_m) + p_{t,1}(\omega_m))z + \ldots + p_{t,l}(\omega_m)z^l$$

Example 3.3. Let us use the example in the Remark in Chapter 2 that the generating function of the geometric distribution

$$P(X_t = 0) = \alpha, P(X_t = 1) = (1-\alpha)q \ , \ \ldots \ P(X_t = n) = (1-\alpha)qp^{n-1}, \ldots$$

Then

$$Ez^{X_\tau} = \alpha + \frac{(1-\alpha)qz}{1-pz} = \frac{\alpha + z(q-\alpha)}{1-pz}$$

We are going to change $\alpha$ into $\alpha_t, p_t \in (0,1)$. They are independent random variables for different $t = 1, 2, \ldots$ with values on $(0, 1)$ and for simplicity for fixed $t$, the parameters $\alpha_t, p_t$ these are also independent. Then it become

$X_t$ is random variable get values $0, 1, 2, \ldots, n \ldots$ with corresponding probabilities $\alpha_t, (1-\alpha_t)q_t, (1-\alpha_t)q_t p_t, \ldots, (1-\alpha_t)q_t p_t^{n-1}, \ldots$ where $q_t = 1 - p_t$ Then

$$\alpha_t+(1-\alpha_t)(q_t z+q_t p_t z^2+\ldots) = \alpha_t+(1-\alpha_t)\frac{q_t z}{1-p_t z} = \frac{\alpha_t - \alpha_t z + q_t z}{1-p_t z} = \frac{\alpha_t + (q_t - \alpha_t)z}{1-(1-q_t)z}$$

Assume also that $\alpha_t \leq 1-\varepsilon$ and $p_t \leq 1-\varepsilon$ for all $t$ and some non-random $\varepsilon > 0$. This will give the existence of all moments of $X_t$ with good estimations.

The generating functions $\varphi_t(z, \omega)$ in this case will be analytic in the circle $|z| < \frac{1}{1-\varepsilon}$ and uniformly bounded in each circle $|z| < \frac{1}{1-\varepsilon'}$, $\varepsilon' > \varepsilon$.

We will assume in general the same condition

$$|\varphi_t(z, \omega_m| \leq \varphi(z, \omega) \leq c_0$$

where $|z| \leq 1 + \varepsilon_0$, for some $\varepsilon_0 > 0$, i.e (due to Cauchy theorem).

$$EX_{t_j}^l \leq C_0 \cdot l! \cdot \frac{1}{\varepsilon_0^l}$$

Let $\varepsilon < \alpha_t < 1-\varepsilon$ and $\varepsilon < p_t < 1-\varepsilon$. Then the moment generating function becomes

$$\varphi_t(z, \omega_m) = \frac{\alpha_t + (q_t - \alpha_t)z}{1-(1-q_t)z}$$

where $(\alpha_t, q_t) \in (0, 1) \times (0, 1)$ and $\alpha_t, q_t$ are i.i.d random vectors.

### 3.2 Annealed and quenched law

The probability measure $P_m$ that determines the distribution of the $n(t, \omega, \omega_m)$ in a given environment $\omega_m \in \Omega$ is referred to as the quenched law, while the full probability measure $P_m \times P$ is referred to as the annealed law.

Here, and in the future, the symbol $\langle\ \rangle$ or $\langle\ \rangle_{\omega_m}$ means the expectation with respect to the probability measure $P_m$ of the random medium. The notation $E$ or $E_x$ will be used for the expectation over the quenched probability measure $P$ for the random walk, starting from $x$ and the fixed environment $\omega_m$.

### 3.3 Moments and quenched classification of Galton-Watson process in random environment

*Recall.* Let $n(t,\omega,\omega_m)$ be the number of the particles at the moment $t = 0, 1, \ldots$. Then

$$n(t+1,\omega,\omega_m) = \sum_{i=1}^{n(t,\omega,\omega_m)} X_{t,i}(\omega_m), \quad (3.1)$$

where $X_{t,i}(\omega_m)$ is an i.i.d random variable whose distribution depends on $t$ and is selected independently for different generations.

Proposition 3.1.

$$Ez^{n(t,\omega,\omega_m)} = \sum_{\rho=0}^{\infty} z^\rho P\{n(t,\omega,\omega_m) = \rho|\omega_m\} = \varphi_1(\varphi_2(\ldots\varphi_t(z,\omega_m)\ldots)) =$$

$$= \varphi_1 \circ \cdots \circ \varphi_t(z,\omega_m) := \varphi_{[t]}(z,\omega_m) \quad (3.2)$$

*where*

$$\varphi_{t,\cdot}(z,\omega_m) = Ez^{X_t} = \sum_{\rho=0}^{\infty} P(X_t = \rho, |\omega_m) z^l \quad (3.3)$$

Generating function of $n(t,\omega,\omega_m)$ is

$$Ez^{n(t,\omega,\omega_m)} = \psi_t(z) = \psi_{t-1}(\varphi_t(z)) \quad (3.4)$$

Let us differentiate the above generating function (3.4) with respect to $z$, at $z = 1$. This will give

$$N_t(\omega_m) = E[n(t,\omega,\omega_m)|\omega_m] = a_1(\omega_m) \cdot \ldots \cdot a_t(\omega_m) \quad (3.5)$$

where

$$a_t(\omega_m) = \frac{d\ \varphi_t}{dz}(z, \omega_m)|_{z=1} = \sum_{\rho=0}^{\infty} \rho P_t(\rho, \omega_m) \tag{3.6}$$

Mean number, first moment of the offspring in the $t^{th}$ generation:

$$N_t(\omega) = e^{\sum_{\rho=1}^{t} \ln a_\rho(\omega_m)} \tag{3.7}$$

There are three cases of quenched classification:

1. Supercritical case

$$\langle \ln\ a(\omega_m) \rangle > 0$$

then by Strong Law of Large Numbers it satisfies:

$$\frac{\ln N_t(\omega_m)}{t} \to \langle \ln\ a_. \rangle$$

2. Critical case

$$\langle \ln\ a(\omega_m) \rangle = 0$$

3. Subcrictical case

$$\langle \ln\ a(\omega_m) \rangle < 0$$

In the last case $P_{w_m}$ is almost surely. $N_t(\omega) \to 0$ if $t \to \infty$ fast and $P_{w_m}$ is a.s

$$P\{n_t \geq 1, \omega_m\} \leq \frac{1}{\langle n_t \rangle} \leq e^{-\delta t}$$

for some $\delta > 0$ and $t \geq t_0(\omega_m)$

The Borel-Cantelli lemma gives for fixed $\omega_m$ and $P_{w_m}$ is almost surely.

$$n(t, \omega, \omega_m) \equiv 0\ , \quad n \geq n_.(\omega_m)$$

In this case the branching process is degenerate.

Theorem 3.2. *Assume that $\langle a_{\cdot}(\omega_m)\rangle = \gamma > 0$, then $P_{\omega_m}$ almost surely for fixed $\omega_m$ the law of the random variable*

$$\frac{n_t(\omega, \omega_m)}{E[n(t, \omega, \omega_m)|\omega_m]} = \frac{n_t(\omega, \omega_m)}{\prod_{s=1}^{t} a_s(\omega_m)} \to n^*(\omega_m)$$

*The random variable $n^*$ has non-trivial distribution with exponentially decreasing tails.*

This is main theorem of Chapter 3. We will prove it by two different ways. First we use the martingale approach, but this method cannot give any information about $n^*$.

*Proof.* Consider

$$\zeta(t, \omega, \omega_m) = \frac{n_t(\omega, \omega_m)}{a_1(\omega_m) \cdot a_2(\omega_m) \cdot \ldots \cdot a_t(\omega_m)}$$

and $F_{\leq t} = \sigma(n_1(\omega_m), \ldots n_t(\omega_m)$

$$E(n(t+1, \omega, \omega_m)|F_{\leq t}(\omega_m)) = n(t, \omega) \cdot a_{t+1}(\omega)$$

$$E(\zeta(t+1, \omega, \omega_m)|F_{\leq t}(\omega_m)) = \frac{n(t, \omega) \cdot a_{t+1}(\omega)}{a_1(\omega_m) \cdot a_2(\omega_m) \cdot \ldots \cdot a_{t+1}(\omega_m)} = \zeta(t, \omega)$$

So $\left(\frac{n(t,\omega,\omega_m)}{\prod_{s=1}^{t} a_s(\omega_m)}, F_{\leq t}(\omega_m)\right)$ is martingale. Due to Doob martingale convergence theorem $p(\omega_m)$ almost surely there exist

$$\lim_{t \to \infty} \zeta(t, \omega, \omega_m) \to \zeta(\infty, \omega_m) \tag{3.8}$$

*Q.E.D.*

Let us find asymptotic formulas for the quenched moments

$$A^{(n)}(t,\omega_m) = E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-n+1)\} \tag{3.9}$$

$$a^{[n]}(t,\omega_m) = EX_t(\omega_m)(X_t(\omega_m)-1)\dots(X_t(\omega_m)-n+1) \tag{3.10}$$

$$\nu_k(t,\omega_m) = \frac{E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-k+1)}{\prod_{s=1}^{t} a_s^k(\omega_m)} \tag{3.11}$$

where

$$E\{X_t(\omega_m)(X_t(\omega_m)-1)\dots(X_t(\omega_m)-n+1)\}$$

and

$$E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-n+1)\}$$

are the factorial moment of the random variable $X_t(\omega_m)$ and $n(t,\omega_m)$ . For simplicity of notation, let us denote the first moment as $a^{[1]}(t,\omega_m) = a_t(\omega_m) = EX_t(\omega_m) > 1$.

Theorem 3.3.

$$\nu_k(t,\omega_m) = \frac{E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-k+1)}{\prod_{s=1}^{t} a_s^k(\omega_m)} \to \nu_k(\omega_m) \tag{3.12}$$

*and* $\nu_k(\omega_m) \leq C_1(\omega_m)k!C^k(\omega_m)$ *for appropriate* $C_1, C > 0$. *Calculation of the limiting moments can be done recursively.*

*Proof.* Let us proof Theorem (3.3).

$$Ez^{n(t,\omega,\omega_m)} = \psi_t(z) = \psi_{t-1}(\varphi_t(z)) = \varphi_1 \circ \dots \circ \varphi_t(z,\omega_m) \tag{3.13}$$

Let us find the second moment of $n(t,\omega,\omega_m)$ random variable.

Differentiating two times of (3.13) gives us

$$\begin{aligned} E\{n(t,\omega_m)[n(t,\omega_m)-1]z^{n(t,\omega_m)-2}\} &= \psi''_{t-1}(\varphi(z,\omega_m))\cdot(\varphi'_t(z,\omega_m))^2 \\ &+\psi'_{t-1}(\varphi(z,\omega_m))\varphi''_t(z,\omega_m) \end{aligned} \tag{3.14}$$

When $z=1$ in (3.14) it gives us second order factorial moment

$$\begin{aligned} &E\{n(t,\omega_m)[n(t,\omega_m)-1]\} = \psi''_{t-1}\cdot(\varphi'_t)^2+\psi'_{t-1}\varphi''_t \\ &= E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}\cdot a_t^2(\omega_m) \\ &+E[n(t-1,\omega_m)]E[X_t(\omega_m)(X_t(\omega_m)-1)] \end{aligned}$$

Let's divide second factorial moment to $\prod_{s=1}^{t-1} a_s^2(\omega_m)$ then we will get

$$\begin{aligned} \frac{E\{n(t,(\omega_m)[n(t,(\omega_m)-1]\}}{\prod_{s=1}^{t} a_s^2(\omega_m)} &= \frac{E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}}{\prod_{s=1}^{t-1} a_s^2(\omega_m)} \\ &+\frac{a^{[2]}(t,\omega_m)}{a_1(\omega_m)\cdot\ldots\cdot a_{t-1}(\omega_m)a_t^2(\omega_m)} \end{aligned}$$

By induction we will get

$$\begin{aligned} \frac{E\{n(t,\omega_m)[n(t,\omega_m)-1]\}}{\prod_{s=1}^{t} a_s^2(\omega_m)} &= \frac{a^{[2]}(t,\omega_m)}{a_1(\omega_m)\cdot\ldots\cdot a_{t-1}(\omega_m)a_t^2(\omega_m)} \\ &+ \frac{a^{[2]}(t-1,\omega_m)}{a_1(\omega_m)\cdot\ldots\cdot a_{t-2}(\omega_m)a_{t-1}^2(\omega_m)}+\ldots+\frac{a^{[2]}(1,\omega_m)}{a_1^2(\omega_m)} \end{aligned}$$

Let us write it a different way

$$\frac{E\{n(t,\omega_m)[n(t,\omega_m)-1]\}}{\prod_{s=1}^{t} a_s^2(\omega_m)} = \sum_{k=1}^{t} \frac{a^{[2]}(k,\omega_m)}{\left(\prod_{l=1}^{k} a_l(\omega_m)\right) a_k(\omega_m)} \tag{3.15}$$

Lemma 3.4. *The*(3.15) *last series converges and has non-random estimation from above.*

Second factorial moment becomes

$$E\{n(t,\omega_m)[n(t,\omega_m)-1]\} = \sum_{k=1}^{t} \frac{a^{[2]}(k,\omega_m)}{\left(\prod_{l=1}^{k} a_l(\omega_m)\right) a_k(\omega_m)} \cdot \prod_{s=1}^{t} a_s^2(\omega_m) \tag{3.16}$$

Let us now find the equation of the third moment. In order to do that we need to differentiate three times and we will get

$$\begin{aligned} E\{n(t,\omega_m)[n(t,\omega_m)-1][n(t,\omega_m)-2]z^{n(t,\omega_m)-2}\} &= \psi'''_{t-1}(\varphi(z)) \cdot (\varphi'_t(z))^3 \\ +3\psi''_{t-1}(\varphi(z))\varphi'_t(z)\varphi''_t(z) &+ \psi'_{t-1}(\varphi(z))\varphi'''_t(z) \end{aligned} \tag{3.17}$$

When $z = 1$ in (3.17) becomes

$$\begin{aligned} &E\{n(t,\omega_m)[n(t,\omega_m)-1][n(t,\omega_m)-2]\} = \psi'''_{t-1} \cdot (\varphi'_t)^3 + 3\psi''_{t-1}\varphi'_t\varphi''_t + \psi'_{t-1}\varphi'''_t \\ &= E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1][n(t-1,\omega_m)-2]\} \cdot a_t^3(\omega_m) \\ &+3E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}a_t(\omega_m)E[X_t(\omega_m)(X_t(\omega_m)-1)] \\ &+E[n(t-1,\omega_m)]E[X_t(\omega_m)(X_t(\omega_m)-1)(X_t(\omega_m)-2)] \end{aligned}$$

$$= E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1][n(t-1,\omega_m)-2]\}\cdot a_t^3(\omega_m)$$

$$+3E\{n(t-1,\omega_m)[n(t-1,\omega_m)-1]\}a_t(\omega_m)a^{[2]}(t,\omega_m)$$

$$+E[n(t-1,\omega_m)]a^{[3]}(t,\omega_m)$$

i.e

$$\frac{E\{n(t,\omega_m)[n(t,\omega_m)-1][n(t,\omega_m)-2]\}}{\prod_{s=1}^{t} a_s^3(\omega_m)} = \nu_3(t-1,\omega_m) + 3\frac{\nu_2(t-1,\omega_m)a^{[2]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s(\omega_m))a_t(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s^2(\omega_m))a_t(\omega_m)}$$

$$= \nu_3(t-1,\omega_m) + 3\frac{a^{[2]}(t,\omega_m)\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{(\prod_{l=1}^{k} a_l(\omega_m))a_k(\omega_m)}}{(\prod_{s=1}^{t} a_s(\omega_m))a_t(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{(\prod_{s=1}^{t} a_s^2(\omega_m))a_t(\omega_m)}$$

$$= \nu_3(t-1,\omega_m) + 3\frac{a^{[2]}(t,\omega_m)\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1]}(\omega_m)a_k(\omega_m)}}{a_{[t,1]}a_t(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{a_{[t,2]}(\omega_m)a_t(\omega_m)}$$

where

$$a_{[t,k]}(\omega_m) = \prod_{s=1}^{t} a_s^k(\omega_m)$$

By induction

$$\nu_3(t,\omega_m) = 3\frac{a^{[2]}(t,\omega_m)\displaystyle\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{(\prod_{l=1}^{k}a_l(\omega_m))a_k(\omega_m)}}{(\displaystyle\prod_{s=1}^{t}a_s(\omega_m))a_t(\omega_m)}$$

$$+3\frac{a^{[2]}(t-1,\omega_m)\displaystyle\sum_{k=1}^{t-2}\frac{a^{[2]}(k,\omega_m)}{(\prod_{l=1}^{k}a_l(\omega_m))a_k(\omega_m)}}{(\displaystyle\prod_{s=1}^{t-1}a_s(\omega_m))a_{t-1}(\omega_m)} + \ldots + 3\frac{\frac{a^{[2]}(2,\omega_m)a^{[2]}(1,\omega_m)}{a_1(\omega_m)a_1(\omega_m)}}{a_1(\omega_m)a_2(\omega_m)a_2(\omega_m)}$$

$$+\frac{a^{[3]}(t,\omega_m)}{(\displaystyle\prod_{s=1}^{t}a_s^2(\omega_m))a_t(\omega_m)} + \frac{a^{[3]}(t-1,\omega_m)}{(\displaystyle\prod_{s=1}^{t-1}a_s^2(\omega_m))a_{t-1}(\omega_m)} + \ldots + \frac{a^{[3]}(1,\omega_m)}{a_1^3(\omega_m)}$$

$$= \quad 3\frac{a^{[2]}(t,\omega_m)\displaystyle\sum_{k=1}^{t-1}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1]}(\omega_m)a_k(\omega_m)}}{a_{[t,1]}(\omega_m)a_t(\omega_m)} + 3\frac{a^{[2]}(t-1,\omega_m)\displaystyle\sum_{k=1}^{t-2}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1]}(\omega_m)a_k(\omega_m)}}{a_{[t-1,1]}(\omega_m)a_{t-1}(\omega_m)} +$$

$$\ldots + 3\frac{\frac{a^{[2]}(2,\omega_m)a^{[2]}(1,\omega_m)}{a_1(\omega_m)a_1(\omega_m)}}{a_1(\omega_m)a_2(\omega_m)a_2(\omega_m)} + \frac{a^{[3]}(t,\omega_m)}{a_{[t,2]}(\omega_m)a_t(\omega_m)}$$

$$+\frac{a^{[3]}(t-1,\omega_m)}{a_{[t-1,2]}(\omega_m)a_{t-1}(\omega_m)} + \ldots + \frac{a^{[3]}(1,\omega_m)}{a_1^3(\omega_m)}$$

$$\nu_3(t,\omega_m) = 3\sum_{j=1}^{t}\left(\frac{a^{[2]}(j,\omega_m)\displaystyle\sum_{k=1}^{j-1}\frac{a^{[2]}(k,\omega_m)}{a_{[k,1](\omega_m)}a_k(\omega_m)}}{a_{[j,1]}(\omega_m)a_j(\omega_m)}\right) + \sum_{j=1}^{t}\frac{a^{[3]}(j,\omega_m)}{a_{[j,2]}(\omega_m)a_j(\omega_m)} \tag{3.18}$$

Lemma 3.5. (3.18) *converges and is bounded above.*

The third factorial moment of $n(t,\omega,\omega_m)$ becomes

$$A^{(3)}(t,\omega_m) = E(n(t,\omega_m)n(t-1,\omega_m)n(t-2,\omega_m)) = \nu_3(t,\omega_m)\cdot a_{[t,3]}(\omega_m) \quad (3.19)$$

Assume that we already proved that for $l \leq k-1$

$$\nu_l(t,\omega_m) = \frac{A^{(l)}}{a_{[t,l]}} = \frac{E\{n(t,\omega_m)(n(t,\omega_m)-1)\dots(n(t,\omega_m)-l+1)}{\prod_{s=1}^{t} a_s^l(\omega_m)} \leq C_1(\omega_m)l!C^l(\omega_m) \quad (3.20)$$

and select such $C_1, C > 0$ that recursively we can reconstruct the same estimate for $l = k$

Meanwhile, a higher order version of the chain rule (see e.g. [11] p. 33) states that

$$f^{(s)}(g) = \sum_{m_1,m_2,\dots,m_s} \frac{s!}{m_1!m_2!\cdots m_s!}[f^{(m_1+m_2+\dots+m_s)}\prod_{j=1}^{s}\left(\frac{g^{(j)}}{j!}\right)^{m_j}]$$

where $1\cdot m_1+\ldots+s\cdot m_s = s$ Let us find the $k^{th}$ derivative. In order to do this, we need to differentiate $k$ times and we will get

$$\begin{aligned}
\psi_t^{(k)} &= \sum_{m_1,m_2,\dots,m_k} \frac{k!}{m_1!m_2!\cdots m_k!}[\psi_{t-1}^{(m_1+m_2+\dots+m_k)}\prod_{j=1}^{k}\left(\frac{\varphi_t^{(j)}}{j!}\right)^{m_j}] \\
&= \psi_{t-1}^{(k)}(\varphi_t')^k + \\
&+ \sum_{\substack{m_1,m_2,\dots,m_k \\ m_1\neq k}} \frac{k!}{m_1!m_2!\cdots m_k!}[\psi_{t-1}^{(m_1+m_2+\dots+m_k)}\prod_{j=1}^{k}\left(\frac{\varphi_t^{(j)}}{j!}\right)^{m_j}]
\end{aligned}$$

where $1 \cdot m_1 + \ldots + k \cdot m_k = k$

$$A^{(k)}(t, \omega_m) = A^{(k)}(t-1, \omega_m) a_t^k +$$

$$+ \sum_{\substack{m_1, m_2, \ldots, m_k \\ m_1 \neq k}} \frac{k!}{m_1! m_2! \cdots m_k!} \left[ (A^{(m_1+m_2+\cdots+m_k)}(t-1, \omega_m) \prod_{j=1}^{k} \left( \frac{a^{(j)}(t, \omega_m)}{j!} \right)^{m_j} \right]$$

where $1 \cdot m_1 + \ldots + n \cdot m_k = k$ and highest order derivative in the last sum is $k-1$ and $\varphi^{(j)}(1) \leq M_1(\omega_m) j! q_1^j(\omega_m)$ and by inductive assumption

$$\psi_t^{(l)}(1) \leq C_1(\omega_m) l! C^l(\omega_m) \prod_{s=1}^{t} a_s^l(\omega_m) \tag{3.21}$$

*Q.E.D.*

# CHAPTER 4: PERCOLATION ON THE RANDOM TREE

## 4.1 Introduction and notation

Suppose we immerse a large porous stone in a bucket of water. What is the probability that the centre of the stone is wetted? In formulating a simple stochastic model for such a situation Broadbent and Hammerslye (1957) gave birth to the percolation model. See more details [12]. In this section we shall establish the basic definitions and notation of bond percolation on tree graphs. A graph is a pair G = (V, E), where V is a set of vertices and B is called the edge or the bond set. If this is finite for each vertex, we call the graph locally finite.

A path in a graph is a sequence of vertices where each successive pair of vertices is an edge in the graph; it is said to join its first and last vertices. A finite path with at least one edge and whose first and last vertices are the same is called a cycle. A graph is connected if for each pair of different vertices there exists a path. A graph with no cycles is called a forest; a connected forest is a tree. Our trees will usually be rooted, meaning that some vertex is designated as the root, denoted o. We imagine the tree as growing (upwards) away from its root. Each vertex then has branches leading to its children, which are its neighbors that are further from the root [16].

Suppose that we close or remove edges at random from a tree with probability $p$, $0 \leq p \leq 1$. We declare this edge to be open with probability $p$ and closed otherwise, independent of all other edges. A principal quantity of interest is the percolation probability $\theta(p)$ being the probability that a given vertex belongs to an infinite open cluster. By the translation invariance of tree lattice and probability measure, we lose

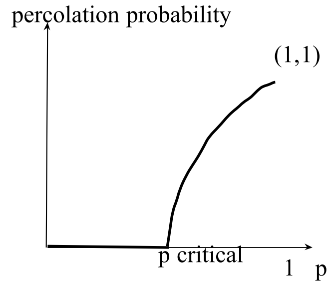


Figure 4.1: The percolation probability $\theta(p)$ behaves roughly as indicated

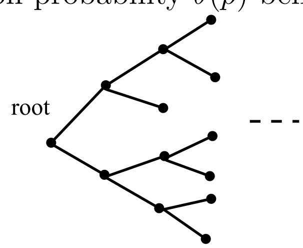


Figure 4.2: Homogeneous binary tree

no generality by taking this vertex to be the origin, thus we define

$$\theta(p) = P_o(|C| = \infty) = 1 - \sum_{n=1}^{\infty} P_o(|C| = n). \tag{4.1}$$

Clearly $\theta$ is a non-decreasing function of $p$ with $\theta(0) = 0$ and $\theta(1) = 1$. It is fundamental to percolation theory that there exists a critical value $p_c$ of $p$ such that [12]

$$\theta(p) = \begin{cases} = 0 & \text{if } p \leq p_c; \\ > 0 & \text{if } p > p_c \ . \end{cases}$$

See figure 4.1 for a sketch of the function $\theta$.

For example if a tree is binary and every edge is either open or closed with probability $p$; then probability of percolation will be $\frac{1}{2}$.

$$\theta(p) = \begin{cases} 0 & \text{if } p < \frac{1}{2}; \\ 1 - \left(\frac{q}{p}\right)^2 & \text{if } p \geq \frac{1}{2} \ . \end{cases}$$

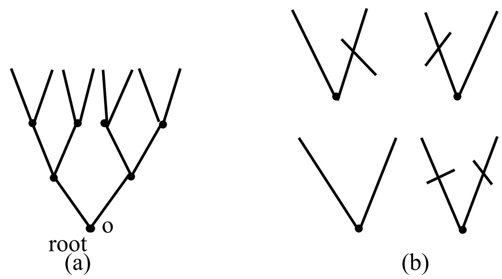


Figure 4.3: Percolation on the binary tree and the binary tree
a) Binary tree with random environment
b) Bond percolation on the binary tree

As we see, by Kolmogorovs 0-1 law, the probability that an infinite connected component remains in the tree is either 0 or 1. On the other hand, we will see that this probability is monotonic in $p$, whence there is a critical value $p_c(T)$ where it changes from 0 to 1. It is also intuitively clear that the bigger the tree, the more likely it is that there will be an infinite component for a given $p$:

Theorem 4.1. *[16]*
*For any tree,* $p_c(T) = \frac{1}{brT}$. *Where* $brT$ *is index of branching of* $T$.

## 4.2 Binary Random tree percolation

Let us consider instead of fixed $p$ we change into $p(\omega_m)$. What will happen to percolation? We consider an infinite directed connected binary tree and it is growing away from the root, see Figure 4.2.

We may consider a binary tree and every edge is either open or closed with probability $p(n, i, \omega_m) = p_{n,i}(\omega_m)$ and $q(n, i, \omega_m) = q_{n,i}(\omega_m) = 1 - p_{\omega_m}(n, i, \omega)$ and this happens independently of the other edges. We may say that our random environment depends on time, space and medium $\omega_m$. At the $n$th level of the tree we have $2^n$ vertexes and edges and so $p_{n,i}(\omega_m)$ where $i \in [1, 2..., 2^n]$,$n = 1, 2....$

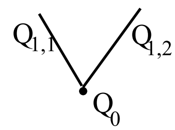


Figure 4.4: Binary tree with random environment
Where $Q_{i,j}(\omega_m) = P(C(i,j,\omega) = \infty)$
is probability that infinite cluster contains root $(i, j)$

Theorem 4.2. *Let $\langle\ \rangle$ or $\langle\ \rangle_{\omega_m}$ be the total expectation on $\omega_n$ and $C_0(\omega, \omega_m)$ is cluster contains root o.*

$$P_{\omega_m}\{|C_0(\omega,\omega_m)| < \infty\} = \begin{cases} 1 & \textit{if } \langle p_{ij}\rangle_{\omega_m} \leq \frac{1}{2}; \\ \rho(\omega_m) & \textit{if } \langle p_{ij}\rangle_{\omega_m} > \frac{1}{2}. \end{cases}$$

*and $\rho(\omega_m)$ is random variable and it is less than 1, $\rho(\omega_m) < 1$.*

*Proof.* Let's fix $\omega_m$ then you can define new random variable

$$Q_{i,j}(\omega_m) = P(\text{There exist infinite cluster containing root } i, j \\ \text{where } i = 1, 2...n, j = 1, ..., 2^n)$$

$Q_{i,j}(\omega_m)$, $\theta(\omega, \omega_m)$ is random variable and it is equal to probability that is infinite cluster contains the root 0.

This can be written

$$\theta(\omega,\omega_m) = P_{\omega_m}(|C_0(\omega,\omega_m)| = \infty) = Q_0(\omega_m)$$

$$
\begin{aligned}
Q_0(\omega_m) =& q_{1,1}(\omega_m)q_{1,2}(\omega_m)0 + p_{1,1}(\omega_m)q_{1,2}(\omega_m)Q_{11}(\omega_m) + p_{1,2}(\omega_m)q_{1,2}(\omega_m)Q_{1,2}(\omega_m) \\
&+ p_{1,1}(\omega_m)p_{1,2}(\omega_m)[1-(1-Q_{1,1}(\omega_m))(1-Q_{1,2}(\omega_m)])
\end{aligned}
$$

where

$$Q_0(\omega_m) \stackrel{law}{=} Q_{1,1}(\omega_m) \stackrel{law}{=} Q_{1,2}(\omega_m)$$

and $\langle Q_0\rangle_{\omega_m} = m_1$

Also $\langle p_0\rangle_{\omega_m} = \langle p_{1,1}\rangle_{\omega_m} = \langle p_{1,2}\rangle_{\omega_m} = 1 - \langle q_{1,1}\rangle_{\omega_m} = 1 - \langle q_{1,2}\rangle_{\omega_m} = 1 - \langle q_0\rangle_{\omega_m}$

Let us take total expectation

$$m_1 = 2\langle p_0\rangle_{\omega_m}\langle q_0\rangle_{\omega_m} m_1 + \langle p_0^2\rangle_{\omega_m}(2m_1 - m_1^2) \tag{4.2}$$

In general

$$
\begin{aligned}
Q_{i,j}(\omega_m) = q_{i+1,2^j-1}(\omega_m)q_{i+1,2^j-1}(\omega_m)0 + p_{i+1,2^j-1}(\omega_m)q_{i+1,2^j}(\omega_m)Q_{i+1,2^j-1}(\omega_m) \\
+ p_{i+1,2^j}(\omega_m)q_{i+1,2^j}(\omega_m)Q_{i+1,2^j}(\omega_m) \\
+ p_{i+1,2^j-1}(\omega_m)p_{i+1,2^j}(\omega_m)[1-(1-Q_{i+1,2^j-1}(\omega_m))(1-Q_{i+1,2^j}(\omega_m)])
\end{aligned} \tag{4.3}
$$

where

$$Q_{i,j}(\omega_m) \stackrel{law}{=} Q_{i+1,2^j-1}(\omega_m) \stackrel{law}{=} Q_{i+1,2^j}(\omega_m)$$

and $\langle Q_{i,j}(\omega_m)\rangle = m_1$

Also

$$\begin{aligned}\langle p_{i,j}\rangle_{\omega_m} &= \langle p_{i+1,2^j-1}\rangle_{\omega_m} = \langle p_{i+1,2^j}\rangle_{\omega_m} = 1-\langle q_{i+1,2^j-1}\rangle_{\omega_m} \\ &= 1-\langle q_{i+1,2^j}\rangle_{\omega_m} = 1-\langle q_{i,j}\rangle_{\omega_m}\end{aligned}$$

Let us now take total expectation

$$m_1 = 2\langle p_{i,j}\rangle_{\omega_m}\langle q_{ij}\rangle_{\omega_m} m_1 + \langle p_{i,j}^2\rangle_{\omega_m}(2m_1 - m_1^2)$$

$$m_1 = 2\langle p_{i,j}\rangle_{\omega_m} m_1 - \langle p_{i,j}\rangle_{\omega_m}^2 m_1^2$$

Hence

$$m_1 = \frac{2\langle p_{i,j}\rangle_{\omega_m} - 1}{\langle p_{i,j}\rangle_{\omega_m}^2}$$

*Q.E.D.*

Theorem 4.3. $|C_0(\omega,\omega_m)|$ *is the volume of the cluster containing a root.*

$$\langle E_{\omega,\omega_m}|C_0(\omega,\omega_m)|\rangle < \infty \qquad \textit{iff } \langle p_{ij}\rangle < \frac{1}{2} \tag{4.4}$$

*Proof.* Let us put $C_0^V(\omega,\omega_m)$ = number of bonds in the connected cluster containing root at the time $\leq V$. Truncated population up to the level $V$
$C^{'V-1}(\omega,\omega_m)$= Number of bonds in the connected cluster containing root $(1,1)$
$C^{''V-1}(\omega,\omega_m)$= Number of bonds in the connected cluster containing root $(1,2)$ and it is illustrated at Figure 4.5.

$$\begin{aligned}C^V(\omega,\omega_m) &= 0\cdot(1-p_{1,1})(1-p_{1,2}) + (1+C^{'V-1}(\omega,\omega_m))p_{1,1}(1-p_{1,2}) \\ &+(1+C^{''V-1}(\omega,\omega_m))(1-p_{1,1})p_{1,2} + (2+C^{''V-1}+C^{'V-1})p_{1,1}p_{1,2}\end{aligned}$$

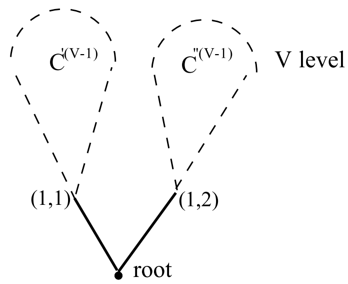


Figure 4.5: Number of bonds in the connected cluster containing root
$C'^{V-1}$= Number of bonds in the connected cluster containing root $(1,1)$
$C''^{V-1}$= Number of bonds in the connected cluster containing root $(1,2)$

Then take expectation and total expectation. Here $\langle p_{i,j}\rangle = \langle p\rangle$

$$\begin{aligned}
\langle EC^V(\omega,\omega_m)\rangle \quad &= \Big(1+\langle EC'^{V-1}(\omega,\omega_m)\rangle\Big)\langle p\rangle(1-\langle p\rangle) \\
&+\Big(1+\langle EC''^{V-1}(\omega,\omega_m)\rangle\Big)(1-\langle p\rangle)\langle p\rangle \\
&+\Big(2+\langle EC''^{V-1}(\omega,\omega_m)\rangle+\langle EC'^{V-1}(\omega,\omega_m)\rangle\Big)\langle p\rangle^2
\end{aligned}$$

Let $m^{(1)}(V) = \langle EC^V(\omega,\omega_m)\rangle$

Then $m(1)(V-1) = \langle EC'^{V-1}(\omega,\omega_m)\rangle = \langle EC''^{V-1}(\omega,\omega_m)\rangle$

$$\begin{aligned}
m(1)(V) \quad &= 2\left(1+m(1)(V-1)\right)\langle p\rangle(1-\langle p\rangle) \\
&+\left(2+2m(1)(V-1)\right)\langle p\rangle^2
\end{aligned}$$

$m(1)(V) = 2\langle p\rangle + 2m(1)(V-1)$ When $V\to\infty$

Answer will be the same, as in the case of the homogeneous percolation, branches with $\widetilde{p} = \langle p\rangle$ probability, and not branches with $1-\widetilde{p} = 1-\langle p\rangle$ probability. In the homogenous binary tree it is well know result.

$$\lim_{V\to\infty}\langle EC^V(\omega,\omega_m)|\rangle < \infty \qquad \text{iff } \langle p\rangle < \frac{1}{2}$$

*Q.E.D.*

Furthermore homogenous percolation cluster is equal to the total number of particles in all generations of the Galton-Watson process

$$X_{\cdot} = \begin{cases} 2 & \text{with probability } \widetilde{p} \\ 0 & \text{with probability } 1-\widetilde{p} \end{cases}$$

and $\widetilde{p}$ get values on $(0,1)$

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